

Homework 6

Math 121A, Fall 2025.

Due: Wednesday, Oct 8, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

- Problem 1.** (a) Choose the branch of the logarithm with range $-\pi \leq \operatorname{Im} \log z < \pi$ (this is called the principal branch with respect to the cut $\mathbb{C} - \mathbb{R}_{\leq 0}$, where we have cut out the negative real axis). What is $\log i$?
- (b) Choose the branch of the logarithm with range $\pi \leq \operatorname{Im} \log z < 3\pi$ (same cut as (a), but different branch). What is $\log i$?
- (c) Calculate i^i separately using both branches.

- Problem 2.** (a) Problem 2.5.1
- (b) Problem 2.5.7
- (c) Problem 2.5.34
- (d) Problem 2.5.50

- Problem 3.** (a) Problem 2.14.21 (*Hint: you may want to write $-1 = e^{i(\pi + k2\pi)}$, $k \in \mathbb{Z}$.*)
- (b) Problem 2.14.24

- Problem 4.** Compute the radius of convergence R for each of the following power series, and assess the convergence on the boundary circle $|z| = R$.

- (a) $\sum nz^n$.
- (b) $\sum \frac{z^n}{n^2}$.
- (c) $\sum \frac{z^n}{n}$.
- (d) $\sum n!z^n$.

- Problem 5.** True or false: $|\sin(z)| \leq 1$ for all $z \in \mathbb{C}$.

- Problem 6.** Express $2e^{-i\pi/2}$ in $x + iy$ form.

- Problem 7.** Find all solutions to $z^3 = i$.

- Problem 8.** Let $z = x + iy$ be a complex number. We may treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$ by choosing the basis $\{1, i\}$, whence the matrix form of z is $z = \begin{bmatrix} x \\ y \end{bmatrix}$.

- (a) Show that the “left-multiplication by z map” $\phi_z : \mathbb{C} \rightarrow \mathbb{C}$

$$\phi_z(w) = zw,$$

is real linear. (Don’t overthink it).

- (b) Since ϕ_z is real linear and we have a real basis for $\mathbb{C}$, we may represent this map by a 2×2 matrix, call it $[\phi_z]$. Compute this matrix.
- (c) Let z_1, z_2 be two complex numbers. Show that $\phi_{z_1 z_2} = \phi_{z_1} \circ \phi_{z_2}$.
- (d) Compute the determinant $\det[\phi_z]$. What is this quantity geometrically?
- (e) Show that $[\phi_z]$ is an orthogonal matrix if and only if $|z| = 1$, i.e. z is on the unit circle.¹
- (f) Now let z be any nonzero complex number. Show that $\frac{1}{\sqrt{\det(z)}}[\phi_z]$ is an orthogonal matrix. What is its determinant? What does this mean geometrically?
- (g) Using what we have learned, interpret the following decomposition geometrically:

$$[\phi_z] = \sqrt{\det(z)} \cdot \frac{1}{\sqrt{\det(z)}}[\phi_z].$$

How does this relate to the polar form of a complex number?

Problem 9. Let’s think a bit about complex conjugation, $f(z) = \bar{z}$.

- (a) What does this function do to the following three real lines in $\mathbb{C}$?
- The line $L_R = \{x + i0 : x \in \mathbb{R}\}$ (“the real line”)?
 - The line $L_I = \{0 + iy : y \in \mathbb{R}\}$ (“the imaginary line”)?
 - The line $L_S = \{s(1 + i) : s \in \mathbb{R}\}$?
- (b) Show that the function f is real linear. As in problem 8, choose the basis $\{1, i\}$ of the real vector space $\mathbb{C}$ and write the matrix $[f]$ corresponding to this map.
- (c) Is the matrix $[f]$ orthogonal? What is $\det[f]$?
- (d) Is the function f complex linear? If so, choose any complex basis of $\mathbb{C}$ and write the matrix corresponding to this map.
- (e) Describe in words what f does to the complex plane $\mathbb{C}$.
- (f) Suppose that $g(z)$ is given by a Taylor series

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

with each a_n real, i.e. $\bar{a}_n = a_n$. Show that $g(\bar{z}) = \overline{g(z)}$. What is an example of such a function?

- (g) Find a holomorphic complex function h for which $h(\bar{z}) \neq \overline{h(z)}$.

¹One way to show a matrix is orthogonal is by showing its columns form an orthonormal basis.